Note Regarding MLE and Asymmetries

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Abstract

This document discusses in which case Maximum Likelihood Estimation (MLE) extracts azimuthal moments or azimuthal asymmetries. This document is an extension of an email by A. Miller, `http://hermes.desy.de/cgi-bin/majord-show.cgi?dir=arch.closed/offline-list&msg=msg130408.0.txt`, linked from the HERMES Wiki page on fitting, `http://hermes-wiki.desy.de/index.php/Fitting`.

MLE Azimuthal Moment Extaction

For example, consider fitting a transverse target data set. Let $\boldsymbol{x}$ be a D dimensional vector of relevant kinematic variables. Without loss of generality, we can separate the $(D+2)$-dimensional differential cross-section into angular integrated, polarized and unpolarized portions,

$$\left(\frac{\partial\sigma_\pm}{\partial\boldsymbol{x}\partial\phi\partial\phi_s}\right) = \left(\frac{\partial\sigma}{\partial\boldsymbol{x}}\right)\left[A_{UU}(\boldsymbol{x},\phi) \pm A_{UT}(\boldsymbol{x},\phi,\phi_s)\right]. \tag{1}$$

Consider (target state balanced) data sets $\{\boldsymbol{x}_\pm^{(i)}\}_{i=1}^{N_\pm}$ distributed according to the above cross-section, and fit using MLE and the function

$$p_\pm \propto 1 \pm p_{UT}(\phi,\phi_s) \tag{2}$$

with

$$p_{UT}(\phi,\phi_s) = a\sin(\phi_s) + \ldots. \tag{3}$$

The log-likelihood function is then

$$L = \sum_{i=1}^{N_+} \ln p_+(\boldsymbol{x}_+^{(i)}) + \sum_{i=1}^{N_-} \ln p_-(\boldsymbol{x}_-^{(i)}). \tag{4}$$

Following A. Miller's email, we consider the limit of infinite statistics, where the sums in the MLE converge to the integral

$$L = \int d^D\boldsymbol{x}d\phi d\phi_s \;\left[\sigma_+(\boldsymbol{x})\ln p_+(\boldsymbol{x}) + \sigma_-(\boldsymbol{x})\ln p_-(\boldsymbol{x})\right]. \tag{5}$$

Note the range of the integration in the infinite statistics case corresponds to the maximum possible range of the data in any finite statistics case. Taking the derivative with respect to the parameter a yields

$$\begin{aligned}
0 &= \frac{\partial L}{\partial a} && (6)\\
&= \int d^D\boldsymbol{x}d\phi d\phi_s \;\left[\frac{\sigma_+(\boldsymbol{x},\phi,\phi_s)}{p_+(\boldsymbol{x},\phi,\phi_s)} - \frac{\sigma_-(\boldsymbol{x},\phi,\phi_s)}{p_-(\boldsymbol{x},\phi,\phi_s)}\right]\sin\phi_s && (7)\\
&= \int d^D\boldsymbol{x}d\phi d\phi_s \;\left(\frac{\partial\sigma}{\partial\boldsymbol{x}}\right)\left[\frac{A_{UU}(\boldsymbol{x},\phi)+A_{UT}(\boldsymbol{x},\phi,\phi_s)}{1+p_{UT}(\phi,\phi_s)} - \frac{A_{UU}(\boldsymbol{x},\phi)-A_{UT}(\boldsymbol{x},\phi,\phi_s)}{1-p_{UT}(\phi,\phi_s)}\right]\sin\phi_s && (8)\\
&= 2\int d^D\boldsymbol{x}d\phi d\phi_s \;\left(\frac{\partial\sigma}{\partial\boldsymbol{x}}\right)\Big[A_{UT}(\boldsymbol{x},\phi,\phi_s) - A_{UU}(\boldsymbol{x},\phi)p_{UT}(\phi,\phi_s)\Big]\frac{\sin\phi_s}{1-p_{UT}^2(\phi,\phi_s)}. && (9)
\end{aligned}$$

Integrating over $\boldsymbol{x}$ yields

$$0 \quad = \quad 2\int d\phi d\phi_s \, \left[\langle A_{UT}\rangle - \langle A_{UU}\rangle\, p_{UT}(\phi,\phi_s)\right]\frac{\sin\phi_s}{1-p_{UT}^2(\phi,\phi_s)}, \tag{10}$$

where the average is with respect to the kinematic variables and the angular-integrated cross-section,

$$\langle f(\boldsymbol{x})\rangle = \int d^D\boldsymbol{x}\left(\frac{\partial\sigma}{\partial\boldsymbol{x}}\right) f(\boldsymbol{x}). \tag{11}$$

The considered derivative equal to zero (and all derivatives with respect to parameters in p_{UT} is obtained when

$$p_{UT} = \frac{\langle A_{UT}\rangle}{\langle A_{UU}\rangle}. \tag{12}$$

Note, that the above expression yet retains angular dependence. For example, integrating the about expression to determine the expression for the extracted $\widehat{A}_{UT}^{\sin(\phi-\phi_s)}$ term yields

$$2\widehat{A}_{UT}^{\sin(\phi-\phi_s)} = \int d\phi d\phi_s \;\sin(\phi-\phi_s)\left[\frac{1+\left\langle A_{UT}^{\sin(\phi_s)}\right\rangle\sin(\phi_s)+\left\langle A_{UT}^{\sin(\phi-\phi_s)}\right\rangle\sin(\phi_s-\phi_s)+\ldots}{1+\left\langle A_{UU}^{\cos(\phi)}\right\rangle\cos(\phi)+\left\langle A_{UU}^{\cos(2\phi)}\right\rangle\cos(2\phi)}\right]. \tag{13}$$

Therefore, the extracted $\widehat{A}_{UT}^{\sin(\phi-\phi_s)}$ then has contributions both from $\left\langle A_{UT}^{\sin(\phi-\phi_s)}\right\rangle$ but likewise from the product $\left\langle A_{UT}^{\sin(\phi_s)}\right\rangle\left\langle A_{UU}^{\cos(\phi)}\right\rangle$.

Also note, that in the MLE the integration is preformed before the ratio, so that in general the acceptance does not cancel. Only if the data is from a sufficiently small domain where

$$\frac{\langle A_{UT}\rangle}{\langle A_{UU}\rangle} \approx \left\langle\frac{A_{UT}}{A_{UU}}\right\rangle \tag{14}$$

does if follow

$$p_{UT} \approx \frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-}. \tag{15}$$

Going through the same exercise with

$$p_\pm \propto 1 + p_{UU}(\phi) \pm p_{UT}(\phi,\phi_s) \tag{16}$$

yields the solution

$$p_{UU} \quad = \quad \langle A_{UU}\rangle\,, \tag{17}$$

$$p_{UT} \quad = \quad \langle A_{UT}\rangle\frac{p_{UU}}{\langle A_{UU}\rangle} = \langle A_{UT}\rangle\,. \tag{18}$$

Thus if one fits or correctly inputs the angular portion of unpolarized cross-section, then indeed the amplitudes are extracted. However, if one does not include the unpolarized moments in the fit function, then indeed the asymmetry is extracted, not the amplitudes. Particular note should be made that this statement make no qualifications about how many unpolarized terms must be included, only stating that all non-negligible, unpolarized, angular terms must be included.